

## A DECOMPOSITION THEOREM FOR $BV$ FUNCTIONS

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**ABSTRACT.** The Jordan decomposition states that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is of bounded variation if and only if it can be written as the difference of two monotone increasing functions.

In this paper we generalize this property to real valued  $BV$  functions of many variables, extending naturally the concept of monotone function. Our result is an extension of a result obtained by Alberti, Bianchini and Crippa.

A counterexample is given which prevents further extensions.

**1. Introduction.** One of the necessary and sufficient properties, which characterizes real valued  $BV$  functions of one variable, is the well-known Jordan decomposition: it states that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is of bounded variation if and only if it can be written as the difference of two monotone increasing functions.

The aim of this work is to give a generalization of this property to real valued  $BV$  functions of many variables.

The starting point is a recent result presented in [1], which shows that a real Lipschitz function of many variables with compact support can be decomposed in sum of monotone functions. Precisely the authors give the following definition of monotone function

**Definition 1.1.** A function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ , which belongs to  $\text{Lip}(\mathbb{R}^N)$ , is said to be *monotone* if the level sets  $\{f = t\} := \{x \in \mathbb{R}^N \mid f(x) = t\}$  are connected for every  $t \in \mathbb{R}$ .

and state the theorem below.

**Theorem 1.2.** *Let  $f$  be a function in  $\text{Lip}_c(\mathbb{R}^N)$  with compact support. Then there exists a countable family  $\{f_i\}_{i \in \mathbb{N}}$  of functions in  $\text{Lip}_c(\mathbb{R}^N)$  such that  $f = \sum_i f_i$  and each  $f_i$  is monotone. Moreover there is a pairwise disjoint partition  $\{A_i\}_{i \in \mathbb{N}}$  of  $\mathbb{R}^N$  such that  $\nabla f_i$  is concentrated on  $A_i$ .*

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In the case of  $BV$  functions, which are defined  $\mathcal{L}^N$ -a.e., an appropriate generalization of the concept of monotone function has to involve super-level sets, sub-level sets and the concept of indecomposable set, as given in [2].

**Definition 1.3.** A set  $E \subseteq \mathbb{R}^N$  with finite perimeter is said to be *decomposable* if there exists a partition  $(A, B)$  of  $E$  such that  $P(E) = P(A) + P(B)$  and both  $|A|$  and  $|B|$  are strictly positive. A set  $E$  is said to be *indecomposable* if it is not decomposable.

Here and in the following  $|E|$  means the Lebesgue measure of the set  $E$ , for  $E$  measurable.

**Definition 1.4.** A function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ , which belongs to  $L^1_{loc}(\mathbb{R}^N)$ , is said to be *monotone* if the super-level sets  $\{f > t\} := \{x \in \mathbb{R}^N \mid f(x) > t\}$  and the sub-level sets  $\{f < t\} := \{x \in \mathbb{R}^N \mid f(x) < t\}$  are of finite perimeter and indecomposable for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$ .

As proved in Section 3, in the case of Lipschitz functions, Definition 1.1 and Definition 1.4 are equivalent.

Other definitions of monotone function can be given.

One can in fact preserve the monotonicity of the product  $\langle f(x) - f(y), x - y \rangle \geq 0$  for  $x \in \mathbb{R}$ , defining that  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is monotone if

$$\langle f(x) - f(y), x - y \rangle \geq 0,$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^N$ .

Another possibility is to preserve the maximum principle: the supremum (infimum) of  $f$  in every set is assumed at the boundary. Taken  $\Omega \subset \mathbb{R}^N$ , a Lebesgue monotone function is defined as a continuous function  $f : \Omega \rightarrow \mathbb{R}$ , which satisfies the maximum and minimum principles in every subdomain. Manfredi, in [6], and Hajlasz and Malý, in [5], give a weaker formulations. Here, a weakly monotone function is defined as a function  $f : \Omega \rightarrow \mathbb{R}$  in the Sobolev space  $W^{1,p}(\Omega)$ , which satisfies the weak maximum and the weak minimum principles in every subdomain. A natural generalization is given in the case  $f$  is in the Sobolev space  $W^{1,p}_{loc}(\Omega)$ .

In our case we choose to maintain the property that sub/super-level sets are connected. Differences and analogies from the case of functions of one variables arise.

On the one hand, it can be found an  $L^1$  monotone function, which is not of bounded variation, that is a counterexample to the fact that monotonicity is a sufficient condition for being of bounded variation (Example 2).

On the other hand, it can be stated that a  $BV$  function is decomposable in a countable sum of monotone functions, similarly to the case of  $BV$  functions of one real variable.

The main result of the paper is the following.

**Theorem 1.5** (Decomposition Theorem for  $BV$  functions). *Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be a  $BV(\mathbb{R}^N)$  function. Then there exists a finite or countable family of monotone  $BV(\mathbb{R}^N)$  functions  $\{f_i\}_{i \in I}$ , such that*

$$f = \sum_{i \in I} f_i \quad \text{and} \quad |Df| = \sum_{i \in I} |Df_i|.$$

This decomposition is in general not unique, see Remark 2.

The main tool for proving this theorem is a decomposition theorem for sets of finite perimeter, presented here in the form given in [2].

**Theorem 1.6** (Decomposition Theorem for sets). *Let  $E$  be a set with finite perimeter in  $\mathbb{R}^N$ . Then there exists a unique finite or countable family of pairwise disjoint indecomposable sets  $\{E_i\}_{i \in I}$  such that*

$$|E_i| > 0 \quad \text{and} \quad P(E) = \sum_{i \in I} P(E_i).$$

Moreover, denoting with

$$\mathring{E}^M := \left\{ x \in \mathbb{R}^N \mid \lim_{r \rightarrow 0^+} \frac{|E \cap B(x, r)|}{|B(x, r)|} = 1 \right\}$$

the essential interior of the set  $E$ , it holds

$$\mathcal{H}^{N-1} \left( \mathring{E}^M \setminus \bigcup_{i \in I} \mathring{E}_i^M \right) = 0$$

and the  $E_i$ 's are maximal indecomposable sets, i.e. any indecomposable set  $F \subseteq E$  is contained, up to  $\mathcal{L}^N$ -negligible sets, in some set  $E_i$ .

The property stated in Theorem 1.2 (there is a disjoint partition  $\{A_i\}_{i \in \mathbb{N}}$  of  $\mathbb{R}^N$  such that every derivative  $\nabla f_i$  of the decomposition is concentrated on  $A_i$ ) is no longer preserved in the case of  $BV$  functions. Example 1 shows that, in general, this decomposition can generate monotone  $BV$  functions without mutually singular distributional derivatives.

Finally, we conclude the paper showing that there is no hope for a further generalization of this decomposition to vector valued  $BV$  functions, apart from the case of a function  $f : \mathbb{R} \rightarrow \mathbb{R}^m$  where the analysis is straightforward. We consider Lipschitz functions from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  and the related definition of monotone function. In this particular case, we construct a counterexample showing that the decomposition property is not true in general, see Example 3.

In fact, a necessary condition for the decomposability of a Lipschitz function, from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , is that some of its level sets must be of positive  $\mathcal{H}^1$ -measure. This is an additional property, which is clearly not shared by all the Lipschitz functions.

The paper is organized as follows.

In Section 2 we prove the main theorem and show that this decomposition can generate monotone  $BV$  functions without mutually singular distributional derivatives.

In Section 3 we give two counterexamples: the first to the fact that a monotone function is always a  $BV$  function, the second to a further extension of the main theorem to vector valued functions. We also give a proof of the fact that for Lipschitz functions Definition 1.1 and Definition 1.4 are equivalent.

**2. The Decomposition Theorem for  $BV$  functions from  $\mathbb{R}^N$  to  $\mathbb{R}$ .** To generalize the Jordan decomposition property, let us concentrate on functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ , which belong to  $BV(\mathbb{R}^N)$ . From now on  $N > 1$ .

Since we will consider functions of bounded variation, the Definition 1.4 of monotone function becomes the following:

**Definition 2.1.** A  $BV$  function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is said to be *monotone* if the super-level sets  $\{f > t\} = \{x \in \mathbb{R}^N \mid f(x) > t\}$  and the sub-level sets  $\{f < t\} = \{x \in \mathbb{R}^N \mid f(x) < t\}$  are indecomposable, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$ .

Indeed, we recall that, for  $BV$  functions, super-level sets and sub-level sets are of finite perimeter for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$ .

We now prove the main theorem of this paper.

*Proof of Theorem 1.5.* The proof will be given in several steps.

Before entering into details, let us consider the following simple case.

Let  $f = \chi_E$  with  $E \subseteq \mathbb{R}^N$  a decomposable set of finite perimeter such that  $\mathbb{R}^N \setminus E$  is indecomposable. Thanks to the Decomposition Theorem for sets, there exists a unique finite or countable family of pairwise disjoint indecomposable sets  $\{E_i\}_{i \in I}$  such that

$$|E_i| > 0 \text{ and } P(E) = \sum_{i \in I} P(E_i).$$

To see the properties of  $\mathbb{R}^N \setminus E_i$  let us consider the following lemma.

**Lemma 2.2.** *Let  $E$  be a decomposable set of finite perimeter such that  $\mathbb{R}^N \setminus E$  is indecomposable. Let  $\{E_i\}_{i \in I}$  be the family of its indecomposable components given by the Decomposition Theorem for sets. Then  $\mathbb{R}^N \setminus E_i$  is indecomposable for every  $i \in I$ .*

*Proof.* Let  $\hat{i} \in I$  be fixed. Without loss of generality we can relabel  $\hat{i} = 1$ .

By contradiction, suppose  $\mathbb{R}^N \setminus E_1$  be decomposable and let  $\{F_j\}_{j \in J}$  be the family of its indecomposable components given by the Decomposition Theorem for sets.

It holds

$$\mathbb{R}^N \setminus E_1 = (\mathbb{R}^N \setminus E) \cup \bigcup_{i \in I, i \neq 1} E_i \pmod{\mathcal{L}^N},$$

where, we recall,  $(\mathbb{R}^N \setminus E) \cup \{E_i\}_{i \in I, i \neq 1}$  is a family of indecomposable and pairwise disjoint sets.

From the maximal indecomposability of  $\{F_j\}_{j \in J}$  and  $\{E_i\}_{i \in I}$ , it follows that

$$\exists! \hat{j} \in J \text{ s.t. } \mathbb{R}^N \setminus E \subseteq F_{\hat{j}} \pmod{\mathcal{L}^N}$$

and

$$\forall j \in J, j \neq \hat{j}, \exists! i \in I, i \neq 1, \text{ s.t. } F_j = E_i \pmod{\mathcal{L}^N}.$$

We relabel  $\hat{j} = 1$ .

Moreover, we can find two sub-families  $\{E_{i_l}\}_{l \in L}$  and  $\{E_{i_k}\}_{k \in K}$  of  $\{E_i\}_{i \in I}$  such that

$$\{E_i\}_{i \in I} = \{E_{i_l}\}_{l \in L} \cup \{E_{i_k}\}_{k \in K},$$

and

$$F_1 = (\mathbb{R}^N \setminus E) \cup \bigcup_{l \in L} E_{i_l} \pmod{\mathcal{L}^N},$$

$$\forall k \in K \exists! j \neq 1 \in J \text{ s.t. } E_{i_k} = F_j \pmod{\mathcal{L}^N}.$$

Observe that

$$\mathbb{R}^N \setminus F_1 = E_1 \cup \bigcup_{k \in K} E_{i_k} \pmod{\mathcal{L}^N},$$

where  $\{E_1, E_{i_k} \mid k \in K\}$  is precisely the family of indecomposable sets given by the Decomposition Theorem for sets. Therefore

$$P(\mathbb{R}^N \setminus F_1) = P(E_1) + \sum_{k \in K} P(E_{i_k}).$$

On the other hand

$$\begin{aligned} P(\mathbb{R}^N \setminus E_1) &= \sum_{j \in J} P(F_j) \\ &= P(F_1) + \sum_{k \in K} P(E_{i_k}), \end{aligned}$$

thus

$$P(E_1) = P(F_1) + 2 \sum_{k \in K} P(E_{i_k}).$$

This implies

$$\sum_{k \in K} P(E_{i_k}) = \sum_{j \in J, j \neq 1} P(F_j) = 0,$$

i.e.  $\mathbb{R}^N \setminus E_1$  is equal to  $F_1$ , up to  $\mathcal{L}^N$ -negligible sets.

Therefore  $\mathbb{R}^N \setminus E_1$  must be indecomposable.  $\square$

From this lemma, for every  $i \in I$ ,  $E_i$  and  $\mathbb{R}^N \setminus E_i$  are indecomposable. Therefore the functions  $\chi_{E_i}$  are  $BV(\mathbb{R}^N)$  and monotone, so that the decomposition of  $\chi_E$ ,

$$\chi_E = \sum_{i \in I} \chi_{E_i},$$

gives  $|D\chi_E| = \sum_{i \in I} |D\chi_{E_i}|$  as required.

*Step 0.* We can assume without loss of generality that  $f \geq 0$ : in the general case one can decompose  $f^+$  and  $f^-$  separately.

*Step 1.* The sets  $E^t := \{f > t\}$  are of finite perimeter for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ , thanks to the hypothesis that  $f$  is  $BV(\mathbb{R}^N)$  and coarea formula. Therefore, the Decomposition Theorem for sets gives, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ , pairwise disjoint indecomposable sets  $\{E_i^t\}_{i \in I_t}$  such that

$$\left| E^t \setminus \bigcup_{i \in I_t} E_i^t \right| = 0.$$

In particular, the property of maximal indecomposability yields a natural partial order relation between these sets: since  $t_1 \geq t_2$  gives  $E^{t_1} \subseteq E^{t_2}$ , it follows that, for  $\mathcal{L}^1$ -a.e.  $t_1 \geq t_2 \in \mathbb{R}^+$ ,

$$\forall i \in I_{t_1} \exists! i' \in I_{t_2} \text{ s.t. } E_i^{t_1} \subseteq E_{i'}^{t_2} \pmod{\mathcal{L}^N}.$$

Taken a countable dense subset  $\{t_j\}_{j \in J}$  of  $\mathbb{R}^+$ , such that, for all  $j \in J$ , the sets  $E^j := E^{t_j}$  are of finite perimeter, the countable family  $\{E_i^j\}_{j \in J, i \in I_{t_j}}$  can be equipped with the partial order relation

$$E_i^j \leq E_{i'}^{j'} \iff t_j \leq t_{j'}, E_i^j \supseteq E_{i'}^{j'} \pmod{\mathcal{L}^N}.$$

Therefore there exists at least one maximal countable ordered sequence (here we do not need the Axiom of Choice).

Let  $\{E_{i(j)}^j\}_{j \in J}$  one of these maximal countable ordered sequences.

Notice that, once one of these sequences is fixed, the index  $i$  is a function of  $j$ , by the uniqueness of the decomposition  $\{E_i^j\}_{i \in I_{t_j}}$ .

*Step 2.* Define

$$\tilde{f}(x) := \begin{cases} 0 & x \notin \bigcup_{j \in J} E_{i(j)}^j \\ \sup\{t_j \mid j \in J, x \in E_{i(j)}^j\} & \text{otherwise} \end{cases}$$

Clearly  $0 \leq \tilde{f}(x) \leq f(x)$  for all  $x \in \mathbb{R}^N$ . Indeed, the set

$$\{t_j \mid j \in J, x \in E_{i(j)}^j\} \subseteq \{t_j \mid j \in J, x \in E^j\} \quad \forall x \in \mathbb{R}^N,$$

passing to the supremum one has  $\tilde{f}(x) \leq f(x)$  for all  $x \in \mathbb{R}^N$ . Moreover  $f \in L_{loc}^1(\mathbb{R}^N)$  and  $0 \leq \tilde{f} \leq f$  give  $\tilde{f} \in L_{loc}^1(\mathbb{R}^N)$ .

*Step 3.* Fix  $t \in \mathbb{R}^+$  such that  $E^t$  is a set of finite perimeter. Define  $\tilde{E}^t := \{\tilde{f} > t\}$  and let  $E_{i(t)}^t$  the indecomposable component of  $E^t$  which is contained in a set  $E_{i(j)}^j$  of the maximal countable ordered sequence and contains another  $E_{i(j')}^{j'}$ , for certain  $j, j' \in J$ , up to  $\mathcal{L}^N$ -negligible sets. This is possible for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ .

Due to the maximal indecomposability property, one has that

$$E_{i(j')}^{j'} \subseteq E_{i(t)}^t \subseteq E_{i(j)}^j \pmod{\mathcal{L}^N} \quad \forall t_{j'}, t_j,$$

where  $t_{j'} > t > t_j$ .

Notice that, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ , there exists only one of such an  $E_{i(t)}^t$  among all the indecomposable sets  $E_i^t, i \in I_t$ .

We show that  $\tilde{E}^t = E_{i(t)}^t \pmod{\mathcal{L}^N}$ , for  $\mathcal{L}^1$ -a.e.  $t$  in  $\mathbb{R}^+$ , in two steps.

- First we show that  $\tilde{E}^t \subseteq E_{i(t)}^t \pmod{\mathcal{L}^N}$  for  $\mathcal{L}^1$ -a.e.  $t$  in  $\mathbb{R}^+$ .

For  $x \in \tilde{E}^t = \{\tilde{f} > t\}$ , there exist  $j_1 = j_1(x), j_2 = j_2(x)$  such that

$$\tilde{f}(x) > t_{j_1} > t > t_{j_2} \quad \text{and} \quad x \in E_{i(j_1)}^{j_1} \cap E_{i(j_2)}^{j_2}.$$

Since for all  $t_{j_1} > t > t_{j_2}$  it holds

$$E_{i(j_1)}^{j_1} \subseteq E_{i(t)}^t \subseteq E_{i(j_2)}^{j_2} \pmod{\mathcal{L}^N},$$

it follows that for  $\mathcal{L}^N$ -a.e.  $x \in \tilde{E}^t$  it holds  $x \in E_{i(t)}^t$ , hence

$$\tilde{E}^t \subseteq E_{i(t)}^t \pmod{\mathcal{L}^N}.$$

- Next we show the other inclusion up to countably many values of  $t$ .

Observe that set  $E_{i(t)}^t$  is contained in  $\tilde{E}^{t'}$  for all  $t' < t$ . In fact  $x \in E_{i(t)}^t$  implies  $f(x) > t > t_j > t'$  for some  $j \in J$ , hence  $\tilde{f}(x) \geq t_j > t'$ . Thus for every  $t'_n \nearrow t$  one has  $\bigcap_{t'_n < t} \tilde{E}^{t'_n} \supseteq E_{i(t)}^t$ .

Suppose  $|E_{i(t)}^t \setminus \tilde{E}^t| > 0$ : from  $\tilde{E}^t \subseteq E_{i(t)}^t$  it follows

$$0 < \left| \bigcap_{t'_n < t} \tilde{E}^{t'_n} \setminus \tilde{E}^t \right| = |\{\tilde{f} \geq t\} \setminus \tilde{E}^t|$$

and this implies  $|\{\tilde{f} = t\}| > 0$ . This last condition can be satisfied only for a countable number of  $t \in \mathbb{R}^+$ .

Therefore the set of  $t$ 's such that  $E_{i(t)}^t$  does not coincide with  $\tilde{E}^t$  has zero Lebesgue measure, i.e. for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$  the sets  $\tilde{E}^t$  coincide with  $E_{i(t)}^t$  up to  $\mathcal{L}^N$ -negligible sets. Since the property of being indecomposable is invariant up to  $\mathcal{L}^N$ -negligible sets, they are indecomposable.

In the following we will denote with  $\tilde{t}_k$ ,  $k \in K$ , the countable family of values such that

$$H_k := \{\tilde{f} = \tilde{t}_k\}, \quad |H_k| > 0.$$

*Step 4.* The function  $\tilde{f}$  is  $BV(\mathbb{R}^N)$  and has indecomposable super-level sets.

The indecomposability of the super-level sets of  $\tilde{f}$  was proved in the previous step.

Using coarea formula, see for example Theorem 2.93 of [3], we get

$$\begin{aligned} |D\tilde{f}| &= \int_{-\infty}^{+\infty} P(\{\tilde{f} > t\}) dt \\ &= \int_{-\infty}^{+\infty} P(E_{i(t)}^t) dt \\ &\leq \int_{-\infty}^{+\infty} P(E^t) dt \\ &= |Df| < +\infty. \end{aligned}$$

Thus the function  $\tilde{f}$  is  $BV(\mathbb{R}^N)$ .

*Step 5.* Define the function  $\hat{f} := f - \tilde{f}$ . Clearly  $\hat{f}$  is  $BV(\mathbb{R}^N)$ . The aim of the following steps is to show that its total variation satisfies

$$|D\hat{f}| = |Df| - |D\tilde{f}|.$$

Denote with  $E_1^t$  the super-level sets used to generate the function  $\tilde{f}$ : this can be done setting  $i(t) = 1$  for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ .

It has been proved that, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ , one has  $\{\tilde{f} > t\} = E_1^t$ , up to  $\mathcal{L}^N$ -negligible sets, therefore for such  $t$ 's

$$\begin{aligned} P(\{f > t\}) &= \sum_{i \in I_t} P(E_i^t) \\ &= \sum_{i \in I_t, i > 1} P(E_i^t) + P(\{\tilde{f} > t\}). \end{aligned}$$

We would like to show that, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ , for every  $i \in I_t$ ,  $i > 1$ ,  $E_i^t$  is equal, up to  $\mathcal{L}^N$ -negligible sets, to one of the indecomposable components  $\hat{E}_i^{\hat{t}}$  of  $\{\hat{f} > \hat{t}\}$ , where  $\hat{t} = t - \tilde{t}_i$  for a certain  $\tilde{t}_i$ .

The index  $i$  in  $\tilde{t}_i$  refers to the fact that its value varies with the indecomposable component  $E_i^t$ ,  $i \in I_t, i > 1$ .

We prove it in the following three steps.

*Step 6.* Let  $t$  be such that the set  $E^t$  is of finite perimeter and  $\{E_i^t\}_{i \in I_t}$  are its indecomposable components.

Let us prove that there exists a unique  $k \in K$  such that the set  $E_i^t$ ,  $i \in I_t, i > 1$ , is contained in  $H_k$ , up to  $\mathcal{L}^N$ -negligible sets.

The set  $E_i^t$  is indecomposable and  $E_i^t \cap E_1^t = \emptyset$ . Being  $E_1^j \subseteq E_1^t$  for all  $t_j \geq t$ , up to  $\mathcal{L}^N$ -negligible sets, it follows

$$|E_i^t \cap E_1^j| = 0 \quad \forall t_j \geq t.$$

Therefore, from the definition of  $\tilde{f}$ , for  $\mathcal{L}^N$ -a.e.  $x \in E_i^t$  one has  $\tilde{f}(x) \leq t$ .

Again from the indecomposability of  $E_i^t$  and from the fact that  $E_i^t$  is contained in  $\{f > t_j\}$  for all  $t_j \leq t$ , it follows that there exists a unique  $l \in I_{t_j}$  such that,

$$E_i^t \subseteq E_l^j \pmod{\mathcal{L}^N} \quad \text{and} \quad |E_i^t \cap E_m^j| = 0 \quad \forall m \neq l, \quad m \in I_{t_j},$$

for all  $t_j \leq t$ .

If there exists a  $j'$  such that  $|E_i^t \cap E_1^{j'}| = 0$  then

$$\forall t_j, \quad 0 \leq t_{j'} \leq t_j \leq t \quad |E_i^t \cap E_1^{j'}| = 0,$$

on the other hand if there exists a  $j''$  such that  $E_i^t \subseteq E_1^{j''}$ , up to  $\mathcal{L}^N$ -negligible sets, then

$$\forall t_j, \quad 0 \leq t_j \leq t_{j''} \quad E_i^t \subseteq E_1^{j'} \pmod{\mathcal{L}^N}.$$

Thus, being the definition

$$\tilde{f}(x) := \begin{cases} 0 & x \notin \bigcup_{j \in J} E_1^j \\ \sup\{t_j \mid j \in J, x \in E_1^j\} & \text{otherwise} \end{cases}$$

equivalent to

$$\tilde{f}(x) := \inf\{t_j \mid j \in J, x \notin E_1^j\},$$

it follows that, up to  $\mathcal{L}^N$ -negligible subsets of  $E_i^t$ ,  $\tilde{f}|_{E_i^t} = \text{constant}$ , which belongs to  $\{\tilde{t}_k\}_{k \in K}$ .

In particular, we can order the sets  $E_i^t, i \in I_t, i > 1$ , as  $E_{(k,i)}^t$  where

$$\{E_{(k,i)}^t \mid i \in B_k^t\} = \left\{ E_i^t \mid i \in I_t, i > 1, \quad E_i^t \subseteq H_k \pmod{\mathcal{L}^N} \right\}.$$

Note that  $B_k^t$  could be empty for some  $t \in \mathbb{R}^+$ ,  $k \in K$ .

*Step 7.* Let  $\hat{t} > 0$  such that the set  $\hat{E}^{\hat{t}}$  is of finite perimeter and  $\{\hat{E}_i^{\hat{t}}\}_{i \in \hat{I}_{\hat{t}}}$  are its indecomposable components, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ .

Let us prove that there exists a unique  $k \in K$ , such that the set  $\hat{E}_i^{\hat{t}}$  is contained in  $H_k$ , up to  $\mathcal{L}^N$ -negligible sets.

Define

$$\bar{t} := \sup \{0, t_j \mid j \in J, \hat{E}_i^{\hat{t}} \subseteq E_1^j \pmod{\mathcal{L}^N}\}.$$

It follows that

$$f|_{\hat{E}_i^{\hat{t}}} = \hat{f}|_{\hat{E}_i^{\hat{t}}} + \tilde{f}|_{\hat{E}_i^{\hat{t}}} > \hat{t} + \bar{t} > \bar{t}.$$

For every  $t_j$  in the countable dense sequence such that  $\bar{t} < t_j < \bar{t} + \hat{t}$  there exists a unique  $\bar{i} \in I_{t_j}$  such that

$$\hat{E}_i^{\hat{t}} \subseteq E_{\bar{i}}^j \pmod{\mathcal{L}^N}.$$

Due to the indecomposability of  $\hat{E}_i^{\hat{t}}$ , and, for the definition of  $\bar{t}$ , the index  $\bar{i}$  must be greater than 1.

Therefore  $\tilde{f}|_{\hat{E}_i^{\hat{t}}} = \bar{t}$  and  $\bar{t}$  belongs to  $\{\tilde{t}_k\}_{k \in K}$ .

In particular, we can order the sets  $\hat{E}_i^{\hat{t}}, i \in \hat{I}_{\hat{t}}$ , as  $\hat{E}_{(k,i)}^{\hat{t}}$  where

$$\{\hat{E}_{(k,i)}^{\hat{t}} \mid i \in \hat{B}_k^{\hat{t}}\} = \left\{ \hat{E}_i^{\hat{t}} \mid i \in \hat{I}_{\hat{t}}, \hat{E}_i^{\hat{t}} \subseteq H_k \pmod{\mathcal{L}^N} \right\}.$$

Note that  $\hat{B}_k^{\hat{t}}$  could be empty for some  $\hat{t} \in \mathbb{R}^+$ ,  $k \in K$ .



*Step 8.* In this step we prove that, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ ,  $k \in K$  fixed,

$$\{E_{(k,i)}^t \mid i \in B_k^t\} = \{\hat{E}_{(k,i)}^{t-\tilde{t}_k} \mid i \in \hat{B}_k^{t-\tilde{t}_k}\}.$$

Indeed, fix  $i \in B_k^t$

$$\hat{f}|_{E_{(k,i)}^t} = f|_{E_{(k,i)}^t} - \tilde{f}|_{E_{(k,i)}^t} > t - \tilde{t}_k.$$

Let us consider only the  $t$ 's such that the set  $\{\hat{f} > t - \tilde{t}_k\}$  is of finite perimeter. For its indecomposability,  $E_{(k,i)}^t$  must be contained, up to  $\mathcal{L}^N$ -negligible sets, in  $\hat{E}_{(k,i')}^{t-\tilde{t}_k}$  for a unique  $i' \in \hat{I}_{t-\tilde{t}_k}$ .

Take then the set  $\hat{E}_{(k,i')}^{t-\tilde{t}_k}$ :

$$f|_{\hat{E}_{(k,i')}^{t-\tilde{t}_k}} = \tilde{f}|_{\hat{E}_{(k,i')}^{t-\tilde{t}_k}} + \hat{f}|_{\hat{E}_{(k,i')}^{t-\tilde{t}_k}} > \tilde{t}_k + t - \tilde{t}_k = t.$$

For its indecomposability,  $\hat{E}_{(k,i')}^{t-\tilde{t}_k}$  must be contained, up to  $\mathcal{L}^N$ -negligible sets, in  $E_{(k,i'')}^t$  for a unique  $i'' \in I_t, i'' > 1$ . Thus  $i'' = i$  and  $E_{(k,i)}^t = \hat{E}_{(k,i')}^{t-\tilde{t}_k}$ , up to  $\mathcal{L}^N$ -negligible sets.

Hence

$$\{E_{(k,i)}^t \mid i \in B_k^t\} \subseteq \{\hat{E}_{(k,i)}^{t-\tilde{t}_k} \mid i \in \hat{B}_k^{t-\tilde{t}_k}\}.$$

The same argument, reversed, shows that, once  $i' \in \hat{B}_k^{t-\tilde{t}_k}$  is fixed,  $\hat{E}_{(k,i')}^{t-\tilde{t}_k} = E_{(k,i)}^t$ , up to  $\mathcal{L}^N$ -negligible sets, for a certain  $i \in B_k^t$ . Hence

$$\{E_{(k,i)}^t \mid i \in B_k^t\} \supseteq \{\hat{E}_{(k,i)}^{t-\tilde{t}_k} \mid i \in \hat{B}_k^{t-\tilde{t}_k}\}.$$

In an equivalent way, we can also say that, for  $\mathcal{L}^1$ -a.e.  $\hat{t} \in \mathbb{R}^+$ ,  $k \in K$  fixed,

$$\{\hat{E}_{(k,i)}^{\hat{t}} \mid i \in \hat{B}_k^{\hat{t}}\} = \{E_{(k,i)}^{\hat{t}+\tilde{t}_k} \mid i \in B_k^{\hat{t}+\tilde{t}_k}\}.$$

In the following we relabel  $\hat{E}_{(k,i)}^{\hat{t}}$  and  $E_{(k,i)}^{\hat{t}+\tilde{t}_k}$  in order to have

$$\hat{E}_{(k,i)}^{\hat{t}} = E_{(k,i)}^{\hat{t}+\tilde{t}_k} \pmod{\mathcal{L}^N}.$$

*Step 9.* Coarea formula gives

$$\begin{aligned} |Df| &= \int_{-\infty}^{+\infty} P(\{f > t\}) dt \\ &= \int_{-\infty}^{+\infty} \sum_{i \in I_t, i > 1} P(E_i^t) dt + \int_{-\infty}^{+\infty} P(\{\tilde{f} > t\}) dt. \end{aligned}$$

The final steps consist in showing that

$$\int_{-\infty}^{+\infty} \sum_{i \in I_t, i > 1} P(E_i^t) dt = |D\hat{f}|.$$

*Step 10.* Let  $\{\tilde{t}_k \mid k \in K\}$  the countable set of values such that  $|\tilde{f}^{-1}(\tilde{t}_k)| > 0$ .

Step 6 shows that, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$  and for all  $i \in I_t, i > 1$ , there exists a unique  $k \in K$  such that  $\tilde{f}|_{E_i^t} = \tilde{t}_k$ .

For every  $k \in K$ , let  $\{E_{(k,i)}^t \mid i \in B_k^t\}$  be the set of indecomposable components of  $E^t$  such that  $\tilde{f}|_{E_{(k,i)}^t} = \tilde{t}_k, i > 1$ .

Observe that  $\sum_{i \in B_k^t} P(E_{(k,i)}^t)$  are measurable functions of  $t$ , for all  $k \in K$ : indeed we have

$$\begin{aligned} |D((f - \tilde{t}_k)\chi_{H_k})| &= \int_{\tilde{t}_k}^{+\infty} \sum_{\substack{i \in I_t, i > 1 \\ \{f > t\}_i \subseteq \{\tilde{f} = \tilde{t}_k\}}} P(\{f > t\}_i) dt \\ &= \int_{\tilde{t}_k}^{+\infty} \sum_{i \in B_k^t} P(\{f > t\}_i) dt \leq |Df|(\mathbb{R}^N) < +\infty. \end{aligned}$$

Therefore the function  $t \mapsto \sum_{i \in B_k^t} P(E_i^t)$  is integrable for all  $k \in K$ .

Using this notation, we can write

$$\begin{aligned} \int_{-\infty}^{+\infty} \sum_{i \in I_t, i > 1} P(E_i^t) dt &= \int_{-\infty}^{+\infty} \sum_{k \in K} \sum_{i \in B_k^t} P(E_{(k,i)}^t) dt \\ &= \sum_{k \in K} \int_{-\infty}^{+\infty} \sum_{i \in B_k^t} P(E_{(k,i)}^t) dt \\ &= \sum_{k \in K} \int_{-\infty}^{+\infty} \sum_{i \in \hat{B}_k^{t-\tilde{t}_k}} P(\{\hat{f} > t - \tilde{t}_k\}_{(k,i)}) dt \\ &= \sum_{k \in K} \int_{-\infty}^{+\infty} \sum_{i \in \hat{B}_k^{\hat{t}}} P(\{\hat{f} > \hat{t}\}_{(k,i)}) d\hat{t} \\ &= \int_{-\infty}^{+\infty} \sum_{k \in K} \sum_{i \in \hat{B}_k^{\hat{t}}} P(\{\hat{f} > \hat{t}\}_{(k,i)}) d\hat{t}. \end{aligned}$$

From Step 7 it holds

$$\begin{aligned} \hat{E}^{\hat{t}} &= \bigcup_i \{\hat{E}_i^{\hat{t}} \mid i \in \hat{I}_{\hat{t}}\} \\ &= \bigcup_i \bigcup_{k \in K} \{\hat{E}_{(k,i)}^{\hat{t}} \mid \tilde{f}|_{\hat{E}_i^{\hat{t}}} = \tilde{t}_k, i \in \hat{I}_{\hat{t}}\} \\ &= \bigcup_{k \in K} \bigcup_i \{\hat{E}_{(k,i)}^{\hat{t}} \mid i \in \hat{B}_k^{\hat{t}}\}, \end{aligned}$$

we can write

$$\begin{aligned} \int_{-\infty}^{+\infty} \sum_{k \in K} \sum_{i \in \hat{B}_k^{\hat{t}}} P(\{\hat{f} > \hat{t}\}_{(k,i)}) d\hat{t} &= \int_{-\infty}^{+\infty} \sum_{i \in \hat{I}_{\hat{t}}} P(\{\hat{f} > \hat{t}\}_i) d\hat{t} \\ &= \int_{-\infty}^{+\infty} P(\{\hat{f} > \hat{t}\}) d\hat{t} = |D\hat{f}|. \end{aligned}$$

Step 11. Finally we have

$$\begin{aligned} |Df| &= \int_{-\infty}^{+\infty} P(\{f > t\}) dt \\ &= \int_{-\infty}^{+\infty} P(\{\hat{f} > t\}) dt + \int_{-\infty}^{+\infty} P(\{\tilde{f} > t\}) dt \\ &= |D\hat{f}| + |D\tilde{f}|. \end{aligned}$$

Since  $f$  has bounded variation we can iterate this process at most a countable number of times generating the family of functions  $\tilde{f}_l \in BV(\mathbb{R}^N)$ , such that everyone of them has indecomposable super-level sets, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ .

*Step 12.* Let  $\tilde{f} := \tilde{f}_l$  be one of the functions generated in the previous steps.

If  $\{\tilde{f} < t\}$  is indecomposable for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ , then  $\tilde{f}$  is already monotone. Otherwise we must again decompose  $\tilde{f}$ . If we succeed in decomposing  $\tilde{f}$  in a countable sum of monotone BV functions which preserves total variation we are done, since the decomposition of every function of a countable family in a countable family gives at the end a countable family as required.

In that case define  $\tilde{F}^t := \{\tilde{f} < t\}$  and let  $\{\tilde{F}_i^t\}_{i \in I_t}$  be the family of indecomposable sets given by the Decomposition Theorem for sets for  $\mathcal{L}^1$ -a.e.  $t$  in  $\mathbb{R}^+$ .

As for the super-level sets, we equip the family  $\{\tilde{F}_i^j\}_{i \in I_{t_j}}$  with the natural partial order relation

$$\tilde{F}_i^j \leq \tilde{F}_{i'}^{j'} \iff t_j \geq t_{j'}, \tilde{F}_i^j \supseteq \tilde{F}_{i'}^{j'} \pmod{\mathcal{L}^N}$$

and call  $\{\tilde{F}_1^j\}_{j \in J}$  one of the maximal countable ordered sequences.

Define

$$\tilde{f}(x) := \inf\{t_j \mid j \in J, x \in \tilde{F}_1^j\}.$$

As in the previous case, one has that

- $\tilde{f}$  is  $BV(\mathbb{R}^N)$ ,
- $\{\tilde{f} < t\} = \tilde{F}_1^t$  up to  $\mathcal{L}^N$ -negligible sets and for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ ,
- define  $\hat{f} := \tilde{f} - \tilde{f}$  then  $\hat{f}$  is  $BV(\mathbb{R}^N)$  and

$$|D\tilde{f}| = |D\hat{f}| + |D\tilde{f}|.$$

Recall that, for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ ,  $\{\tilde{f} < t\}$  is decomposable and  $\mathbb{R}^N \setminus \{\tilde{f} < t\}$  indecomposable. Since  $\{\tilde{f} < t\} = \bigcup_{i \in I_t} \tilde{F}_i^t$  and  $\{\tilde{f} < t\} = \tilde{F}_1^t$  up to  $\mathcal{L}^N$ -negligible sets, Lemma 2.2 implies that  $\mathbb{R}^N \setminus \{\tilde{f} < t\}$  is indecomposable, hence the super-level set  $\{\tilde{f} > t\}$  is indecomposable for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^+$ . Therefore  $\tilde{f}$  is monotone as required.

Since  $\tilde{f}$  has bounded variation we can iterate this process at most a countable number of times generating the family of monotone functions  $f_i \in BV(\mathbb{R}^N)$ , which satisfies the theorem. □

**Remark 1.** Notice that in Step 10 we have also proved that

$$\hat{f}|_{\bigcup_{k \in K} H_k} = \sum_{k \in K} f|_{H_k} - \tilde{t}_k.$$

**Remark 2.** In general the decomposition of  $f$  in BV monotone functions is not unique as the following example shows.

The function  $f$  in Figure 1 can be decomposed either in the way shown in Figure 2 or in Figure 3.

In the simple case, where  $f$  is the characteristic function of a set of finite perimeter with an indecomposable complementary set, there exists a unique subdivision of  $f$  as a countable sum of BV monotone characteristic functions. Moreover in that case,

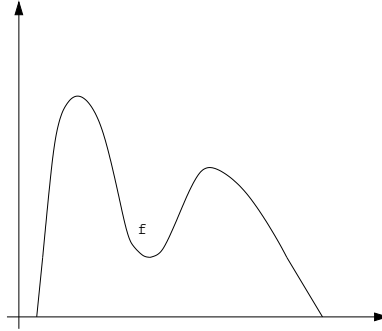
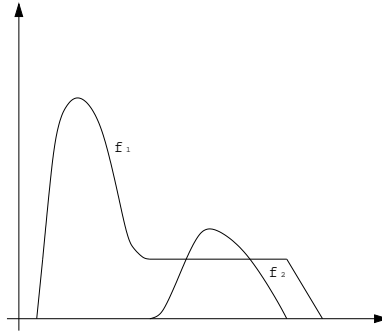
FIGURE 1. Function  $f$ 

FIGURE 2. Decomposition 1

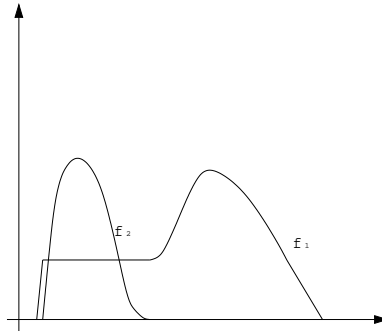


FIGURE 3. Decomposition 2

due to the fact that the sets  $E_i$  are pairwise disjoint,  $D\chi_{E_i}$  are mutually singular for all  $i \in I$ .

This property, which has been proved also for the decomposition of Lipschitz functions in Theorem 1.2, can be false in the general case. As shown in the example below, one can have monotone  $BV$  functions, whose distributional derivatives are concentrated on sets with non empty intersection.

**Example 1.** Let us consider a  $BV$  function  $f$  as in the Figure 4.

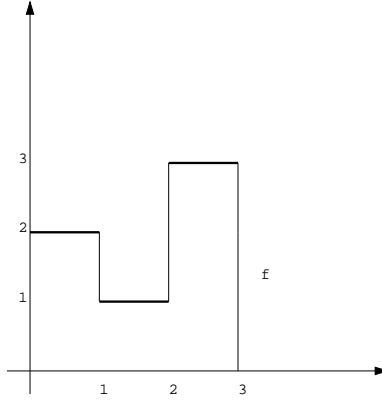
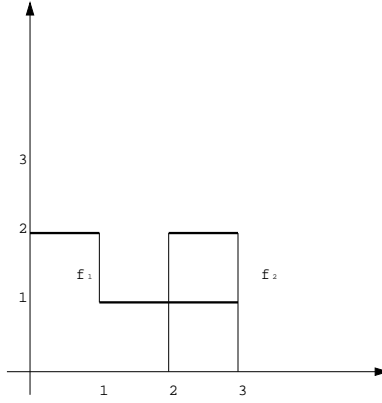
FIGURE 4. Function  $f$ 

FIGURE 5. Decomposition of the function

In this case the Decomposition Theorem gives two  $BV$  monotone functions  $f_1$  and  $f_2$  such that  $f = f_1 + f_2$ . Their distributional derivatives are

$$|Df_1| = 2\delta_0 - \delta_1 - \delta_3 \quad \text{and} \quad |Df_2| = 2\delta_2 - 2\delta_3,$$

where  $\delta_x$  is the Dirac measure,  $\delta_x(A) = 1$  if  $x$  belongs to the set  $A$ ,  $\delta_x(A) = 0$  otherwise. Clearly these distributional derivatives are not mutually singular, since both have an atom in  $x = 3$ .

One can easily show that for any other monotone decomposition it is impossible to find two disjoint sets on which the distributional derivatives are concentrated.

**3. Counterexamples.** As we said in the Introduction, the definition of monotone function could be given even for a function which is only  $L^1_{loc}(\mathbb{R}^N)$ . In that case one has to require that this function must have super-level sets with finite perimeter, which is true  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}$  for the super-level sets of a  $BV$  function.

The Jordan decomposition states that monotonicity is a sufficient condition for a function of one variable to be of bounded variation. However, we cannot say that every monotone function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  defined as in Definition 1.4 is of bounded variation.

A counterexample is given below by a function, whose super-level sets are progressive configurations of the construction of a Koch snowflake.

**Example 2.** The Koch snowflake is a curve generated iteratively from a unitary triangle  $T$  adding each time, on each edge, a smaller centered triangle with edges one third of the previous edge, see Figure 6.

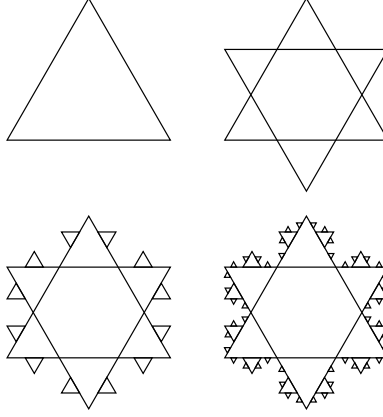


FIGURE 6. Progressive configurations of the construction of a Koch snowflake

More precisely letting  $T_0$  be the equilateral triangle  $T$  with unitary edge, and  $T_i$  the successive iterations of the curve, one has that at every stage

- the number of edges is  $N_k = 3 \cdot 4^k$ ,
- the length of the edges is  $L_k = \left(\frac{1}{3}\right)^k$ ,
- the perimeter of the iterated curve is  $P(T_k) = 3 \cdot \left(\frac{4}{3}\right)^k$ ,
- the area of the iterated curve is

$$|T_k| = \left[ 1 + \frac{1}{3} \sum_{j=1}^k \left(\frac{4}{9}\right)^j \right] \cdot \frac{\sqrt{3}}{2}.$$

Denote with  $B$  the ball

$$B = \{x \in \mathbb{R}^2 \mid \|x\| < R\},$$

which contains the unitary triangle  $T$  centered in the origin: hence  $T_i \subseteq B$  for all  $i \in \mathbb{N}$ .

Let  $E_k := B \setminus T_k$  for  $k \in \mathbb{N}$  and define  $f : B \rightarrow \mathbb{R}$  in this way

$$f(x) := \sum_k \left(\frac{3}{4}\right)^k \chi_{E_k}(x).$$

Clearly  $0 \leq f < 4$ , therefore  $f$  belongs to  $L^1(B)$  and coarea formula can be used to obtain its variation.

Let us note which are the super-level sets and their perimeter:

- for  $t < 0$  the set  $\{f > t\} = B$  and  $P(B, B) = 0$ ,
- for  $t = 0$  the set  $\{f > t\} = E_0$  and  $P(E_0, B) = 3$ ,
- for  $0 < t < 4$  the set  $\{f > t\} = E_{\bar{k}}$  for the first  $\bar{k}$  such that  $\sum_{k=0}^{\bar{k}} \left(\frac{3}{4}\right)^k > t$  and  $P(E_{\bar{k}}, B) = 3 \cdot \left(\frac{4}{3}\right)^{\bar{k}}$ ,

- for  $t \geq 4$  the set  $\{f > t\} = \emptyset$  and  $P(\emptyset, B) = 0$ .

Thus this function is monotone and computing its variation one has

$$\begin{aligned} |Df|(B) &= \int_{-\infty}^{+\infty} P(\{f > t\}, B) dt \\ &= \int_0^4 P(\{f > t\}, B) dt \\ &= \sum_{k=0}^{+\infty} 3 \cdot \left(\frac{4}{3}\right)^k \cdot \left(\frac{3}{4}\right)^k = +\infty \end{aligned}$$

which implies that  $f$  does not belong to  $BV(B)$ .

In the case of Lipschitz functions Definition 1.1 and Definition 1.4 are equivalent.

**Proposition 3.1.** *Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be a Lipschitz function, then  $f$  is monotone in the sense of Definition 1.1 if and only if  $f$  is monotone in the sense of Definition 1.4.*

*Proof.* ( $\Rightarrow$ ) Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be a Lipschitz function which is monotone in the sense of Definition 1.1, then for all  $t$  in  $\mathbb{R}$  the set  $\{f = t\}$  is connected.

We claim that  $\{f > t\}$  and  $\{f < t\}$  are open connected sets. Indeed, let us concentrate on  $\{f > t\}$ , the other case is similar.

By contradiction suppose  $\{f > t\}$  disconnected, then  $\{f > t\}$  must have at least two connected components. For  $t' > t$ , such that  $t' - t$  is sufficiently small, the set  $\{f = t'\}$  is contained at least in two of the connected components of  $\{f > t\}$ . Thus we have a connected set  $\{f = t'\}$  contained in two connected components of a disconnected set, absurd.

Since for  $\mathcal{L}^1$ -a.e.  $t$  in  $\mathbb{R}$  the sets  $\{f > t\}$  and  $\{f < t\}$  are of finite perimeter Proposition 2 in [2] gives that the open and connected sets  $\{f > t\}$  and  $\{f < t\}$  are indecomposable for  $\mathcal{L}^1$ -a.e.  $t$  in  $\mathbb{R}$ .

Therefore  $f$  is monotone in the sense of Definition 1.4.

( $\Leftarrow$ ) Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be a Lipschitz function which is not monotone in the sense of Definition 1.1, then there exists a  $t$  in  $\mathbb{R}$  such that the set  $\{f = t\}$  is disconnected.

For Theorem 6.1.23 in [4], every connected components of  $\{f = t\}$  coincides with a quasi-connected component of  $\{f = t\}$ , because  $\{f = t\}$  is compact.

This implies that there exists an open set  $G$  in  $\mathbb{R}^N$  such that

$$\partial G \cap \{f = t\} = \emptyset, \quad G \cap \{f = t\} \neq \emptyset$$

and

$$(\mathbb{R}^N \setminus \overline{G}) \cap \{f = t\} \neq \emptyset.$$

From its continuity,  $f$  must be greater than  $t$  or lower than  $t$  over the all  $\partial G$ . Let us fix  $f|_{\partial G} < t$ .

The compactness of  $\{f = t\}$  gives the existence of a  $\delta > 0$  such that  $f|_{\partial G} \leq t - \delta$ . Thus, for all  $\varepsilon \in (0, \delta)$ ,

$$\partial G \cap \{f > t - \varepsilon\} = \emptyset \quad \text{and} \quad \{f = t\} \subseteq \{f > t - \varepsilon\}.$$

Therefore

$$G \cap \{f > t - \varepsilon\} \neq \emptyset, \quad (\mathbb{R}^N \setminus \overline{G}) \cap \{f > t - \varepsilon\} \neq \emptyset.$$

In addition, defining  $L$  the Lipschitz constant of  $f$ ,

$$d(\{f \geq t - \varepsilon\}, \partial G) \geq \frac{\delta - \varepsilon}{L}.$$

It follows that the open set  $\{f > t - \varepsilon\}$  can be decomposed into two open sets with positive distance, in particular it is decomposable.

In the case

$$f|_{\partial G} > t,$$

one can similarly show that, for all  $\varepsilon$  in  $(0, \delta)$ , the set  $\{f < t - \varepsilon\}$  is decomposable. Therefore  $f$  is not monotone in the sense of Definition 1.4.  $\square$

The Decomposition Theorem for real valued  $BV$  functions of  $\mathbb{R}^N$  is in some sense optimal. Considering  $BV$  functions from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  one can find counterexamples to this theorem, i.e.  $BV$  functions which cannot be decomposed in sum of  $BV$  monotone functions preserving total variation.

The crucial point is that we require to our decomposition, besides being the sum of  $BV$  monotone functions, to preserve the total variation, i.e.

$$|Df| = \sum_{i \in I} |Df_i|.$$

**Remark 3.** For example, let us generalize as follows our definition of  $BV$  monotone function to functions with values in a space of a greater dimension.

**Definition 3.2.** A function  $f : \mathbb{R}^N \rightarrow \mathbb{R}^m$ , which belongs to  $[BV(\mathbb{R}^N)]^m$ , is said to be *monotone* if the super-level sets

$$\{f > t\} := \{x \in \mathbb{R}^N \mid f_i(x) > t_i \ i = 1, \dots, m\}$$

and the sub-level sets

$$\{f < t\} := \{x \in \mathbb{R}^N \mid f_i(x) < t_i \ i = 1, \dots, m\},$$

are indecomposable, for  $\mathcal{L}^m$ -a.e.  $t \in \mathbb{R}^m$ .

$$\text{Let } f : \mathbb{R}^N \rightarrow \mathbb{R}^m \text{ a } BV \text{ function } f = \begin{pmatrix} f_1 \\ \dots \\ f_m \end{pmatrix}.$$

For  $i = 1, \dots, m$ , every  $f_i$  is a  $BV$  function from  $\mathbb{R}^N$  to  $\mathbb{R}$  so that Theorem 1.5 applies. Therefore, for every  $i = 1, \dots, m$ , one has the decomposition in  $BV$  monotone functions  $f_i = \sum_{j \in J_i} f_i^j$ .

Note that, if  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  is a  $BV$  monotone function, the function  $\begin{pmatrix} 0 \\ \dots \\ g \\ \dots \\ 0 \end{pmatrix}$  is a

$BV$  monotone function too, from  $\mathbb{R}^N$  to  $\mathbb{R}^m$ , in the sense of Definition 3.2.

It follows that we can decompose  $f$  in that way

$$f = \sum_{j \in J_1} \begin{pmatrix} f_1^j \\ 0 \\ \dots \\ 0 \end{pmatrix} + \dots + \sum_{j \in J_m} \begin{pmatrix} 0 \\ \dots \\ 0 \\ f_m^j \end{pmatrix}.$$



However, this decomposition does not preserve the total variation of  $f$  and one can only say that

$$|Df| \leq \sum_{j \in J_1} \begin{pmatrix} |Df_1^j| \\ 0 \\ \dots \\ 0 \end{pmatrix} + \dots + \sum_{j \in J_m} \begin{pmatrix} 0 \\ \dots \\ 0 \\ |Df_m^j| \end{pmatrix}.$$

We give now a counterexample in the case of Lipschitz function from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . In this situation we extend the Definition 1.1.

**Definition 3.3.** A function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , which belongs to  $[\text{Lip}(\mathbb{R}^2)]^2$ , is said to be *monotone* if the level sets  $\{f = t\} = \{x \in \mathbb{R}^2 \mid f(x) = t\}$  are connected for every  $t \in \mathbb{R}^2$ .

We observe that if  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  Lipschitz is a monotone operator, then its level sets are closed convex. Hence the requirement to preserve the connectedness of the level sets is weaker than being a monotone operator.

**Example 3.** Let  $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$  be a Lipschitz function: in this particular case, by area formula it follows that  $f$  is monotone if and only if for  $\mathcal{L}^1$ -a.e.  $t \in \mathbb{R}^2$   $f^{-1}(t)$  is a singleton.

Using Lipschitz continuity, it is simple to verify that if  $f_1 : \mathbb{R}^2 \mapsto \mathbb{R}^2$  is a Lipschitz function such that

$$|Df| = |Df_1| + |D(f - f_1)|,$$

then either  $f = f_1$  or there exists a set with positive length where  $f = f_1$  is constant.

However, not all Lipschitz functions from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  have this particular property. For example consider

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x) = \begin{pmatrix} 1 - \cos(\frac{\pi x_1}{2}) \\ 1 - \cos(\frac{\pi x_2}{2}) \end{pmatrix}.$$

For this function the level sets  $\{f = t\}$  have zero length for every  $t \in \mathbb{R}^2$ . Thus any decomposition with the properties desired is impossible.

	$\mathcal{H}^K$	K-dimensional Hausdorff measure
	$\mathcal{L}^N$	N-dimensional Lebesgue measure
	$\mathbb{R}^+$	set of all non negative real number
	$[L^1(\mathbb{R}^N)]^m$	Lebesgue space of functions from $\mathbb{R}^N$ to $\mathbb{R}^m$
	$L^1_{loc}(\mathbb{R}^N)$	space of functions from $\mathbb{R}^N$ to $\mathbb{R}$ which are locally $L^1(\mathbb{R}^N)$
	$[\text{Lip}_c(\mathbb{R}^N)]^m$	space of c-Lipschitz functions from $\mathbb{R}^N$ to $\mathbb{R}^m$
	$[BV(\mathbb{R}^N)]^m$	space of bounded variation functions from $\mathbb{R}^N$ to $\mathbb{R}^m$
	$\nabla f$	gradient of the Lipschitz function $f$
	$Df$	distributional derivative of the BV function $f$
4. Notations.	$ Df $	total variation of the function $f$
	$P(E)$	perimeter of the set $E$
	$ E $	Lebesgue measure of the set $E$
	$\dot{E}^M$	essential interior of the set $E$
	$\overline{E}$	closure of the set $E$
	$\chi_E$	characteristic function of the set $E$
	$(\text{mod } \mathcal{L}^N)$	up to $\mathcal{L}^N$ -negligible sets
	$\delta_x$	Dirac measure
	$\partial E$	topological boundary of a set $E$
	$d(A, B)$	distance between the sets $A$ and $B$

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